

# Cyclic Orderings and Cyclic Arboricity of Matroids

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## Abstract

We prove a general result concerning cyclic orderings of the elements of a matroid. *For each matroid  $M$ , weight function  $\omega : E(M) \rightarrow \mathbb{N}$ , and positive integer  $D$ , the following are equivalent.* (1) *For all  $A \subseteq E(M)$ , we have  $\sum_{a \in A} \omega(a) \leq D \cdot r(A)$ .* (2) *There is a map  $\phi$  that assigns to each element  $e$  of  $E(M)$  a set  $\phi(e)$  of  $\omega(e)$  cyclically consecutive elements in the cycle  $(1, 2, \dots, D)$  so that each set  $\{e \mid i \in \phi(e)\}$ , for  $i = 1, \dots, D$ , is independent.*

As a first corollary we obtain the following. *For each matroid  $M$  such that  $|E(M)|$  and  $r(M)$  are coprime, the following are equivalent.* (1) *For all non-empty  $A \subseteq E(M)$ , we have  $|A|/r(A) \leq |E(M)|/r(M)$ .* (2) *There is a cyclic permutation of  $E(M)$  in which all sets of  $r(M)$  cyclically consecutive elements are bases of  $M$ .* A second corollary is that the circular arboricity of a matroid is equal to its fractional arboricity.

These results generalise classical results of Edmonds, Nash-Williams and Tutte on covering and packing matroids by bases and graphs by spanning trees.

**Keywords:** matroid, base of a matroid, cyclic ordering, arboricity, circular arboricity

## 1 Introduction and Results

### 1.1 Cyclic Orderings of Matroids

We assume the reader is familiar with the basics of matroid theory, as can be found in, e.g., the book of Oxley [11]. All matroids in this paper are assumed to be finite and without loops. We use  $E$  for the ground set of a matroid under consideration.

The crucial axiom on the bases of a matroid is the *exchange axiom*. There are several forms of this (see [11]); the important one for us is: if  $B$  and  $B'$  are bases and  $b' \in B' \setminus B$ , then there exists  $b \in B \setminus B'$  such that  $(B \setminus \{b\}) \cup \{b'\}$  is a base. This axiom implies that given

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two bases  $B$  and  $B'$ , there exists a sequence of exchanges that transforms  $B$  into  $B'$ , i.e., there is a sequence of bases  $B = B_0, \dots, B_i, \dots, B_r = B'$  for which the symmetric difference of every two consecutive bases has two elements. If  $B$  and  $B'$  are disjoint (hence  $r$  is equal to the rank  $r(M)$ ), one way to get such a sequence is as follows. Let  $B' = \{b'_1, b'_2, \dots, b'_r\}$ . Take  $b_1 \in B$  such that  $(B \setminus \{b_1\}) \cup \{b'_1\}$  is a base, then take  $b_2 \in B$  such that  $(B \setminus \{b_1, b_2\}) \cup \{b'_1, b'_2\}$  is a base, etc. We obtain a sequence  $(b_1, b_2, \dots, b_r, b'_1, b'_2, \dots, b'_r)$  in which every set of  $r$  consecutive elements forms a base (indeed, these can be taken as the sets  $B_i$ ).

Studying the structure of symmetric exchanges in matroids, Gabow [6] asked if it is possible to choose the sequence  $(b_1, \dots, b_r, b'_1, \dots, b'_r)$  so that it provides a *cyclic ordering* in which each  $r$  cyclically consecutive elements form a base; i.e., the sequences  $(b'_2, \dots, b'_r, b_1)$ ,  $(b'_3, \dots, b'_r, b_1, b_2)$ , etc., form bases as well. This question was raised again by Wiedemann [14] and formulated as a conjecture by Cordovil and Moreira [2].

### **Conjecture 1.1 ([2, 6, 14])**

*Let  $B = \{b_1, \dots, b_r\}$  and  $B' = \{b'_1, \dots, b'_r\}$  be two bases of a matroid. There is a permutation  $(b_{\pi(1)}, \dots, b_{\pi(r)})$  of the elements of  $B$  and a permutation  $(b'_{\pi'(1)}, \dots, b'_{\pi'(r)})$  of the elements of  $B'$  such that the combined sequence  $(b_{\pi(1)}, \dots, b_{\pi(r)}, b'_{\pi'(1)}, \dots, b'_{\pi'(r)})$  is a cyclic ordering in which every  $r$  cyclically consecutive elements form a base.*

Conjecture 1.1 has been proved for graphical matroids [2, 8, 14].

A possible easier conjecture is that a suitable cyclic ordering can be obtained by permuting all elements in the union of the two bases.

### **Conjecture 1.2**

*Given two bases  $B = \{b_1, \dots, b_r\}$  and  $B' = \{b'_1, \dots, b'_r\}$  of a matroid, there is a permutation of the sequence  $(b_1, b_2, \dots, b_r, b'_1, b'_2, \dots, b'_r)$  in which every  $r$  cyclically consecutive elements form a base.*

Since Conjecture 1.1 is known to hold for graphical matroids, so does this weaker conjecture.

It is obvious that the linear ordering from the introductory paragraphs exists for any number of bases. Hence a natural generalisation of the previous conjectures is to start with  $k \geq 2$  bases and require a suitable cyclic ordering of the elements of these  $k$  bases combined. No results are known for  $k \geq 3$ , not even for graphical matroids.

Kajitani *et al.* [8] formulated the following much more general conjecture.

### **Conjecture 1.3 ([8])**

*For a loopless matroid  $M$ , there is a cyclic ordering of  $E$  such that every  $r(M)$  cyclically consecutive elements are bases of  $M$  if and only if the following condition is satisfied :*

$$(1) \quad \text{for all non-empty } A \subseteq E, \text{ we have } \frac{|A|}{r(A)} \leq \frac{|E|}{r(M)}.$$

Following Catlin *et al.* [1], matroids that satisfy condition (1) are called *uniformly dense*.

The fact that (1) is necessary for the required cyclic ordering of the ground set to exist was already observed by Kajitani *et al.* [8]. They also proved Conjecture 1.3 for the cycle matroids of some special classes of graphs.

A matroid admitting a partition of its ground set into bases is uniformly dense. Hence Conjecture 1.3 implies Conjecture 1.2 (and the generalisation of this conjecture with an arbitrary number of bases). Conjecture 1.3 also implies that if  $M$  is uniformly dense and  $|E|/r(M) = P/Q$  ( $P, Q \in \mathbb{N}$ ), then there exist  $P$  bases such that each element of  $E$  appears in exactly  $Q$  of them. This weaker result was proved by Catlin *et al.* [1] and Fraisse and Hell [5].

We prove Conjecture 1.3 for a special class of matroids.

#### Theorem 1.4

*Let  $M$  be a loopless matroid such that  $|E|$  and  $r(M)$  are coprime (i.e.,  $\gcd(|E|, r(M)) = 1$ ). There is a cyclic ordering of  $E$  such that every  $r(M)$  cyclically consecutive elements are bases of  $M$  if and only if (1) holds.*

The proof of this theorem can be found in Section 3. It follows from a more technical result we formulate next.

A *weighted matroid*  $(M, \omega)$  is a matroid  $M$  together with a weight function  $\omega : E \rightarrow \mathbb{Q}^+$ . The weight  $\omega(A)$  of a subset  $A \subseteq E$  is the sum of the weights of its elements. For a positive real number  $d$ , let  $S_d$  be the circle with circumference  $d$  (interpreted as the interval  $[0, d]$  with ends identified, or equivalently as the quotient  $\mathbb{R}/d\mathbb{R}$ ). When speaking about a (left-closed, right-open) *cyclic interval*  $[x, y)$  of  $S_d$ , we interpret it as the part of the circle that starts at  $x$  and follows  $S_d$  in the positive direction until reaching  $y$ .

Let  $(M, \omega)$  be a weighted matroid and  $\phi$  a mapping from  $E$  to  $S_d$ . This mapping associates to every  $e \in E$  a cyclic interval  $[\phi(e), \phi(e) + \omega(e))$  of  $S_d$ . Conversely, to every point  $x$  of  $S_d$ , we can associate the set  $E_\phi(x) = \{e \in E \mid x \in [\phi(e), \phi(e) + \omega(e))\}$ .

We are interested in mappings  $\phi : E \rightarrow S_d$  such that  $E_\phi(x)$  is independent for every point  $x$  in  $S_d$ . It is obvious that for large enough  $d$  such a mapping always exists. Our main result gives the exact lower bound on  $d$  for which such a mapping is possible.

#### Theorem 1.5

*Let  $(M, \omega)$  be a loopless weighted matroid and  $d$  a positive rational number. There exists a mapping  $\phi : E \rightarrow S_d$  such that  $E_\phi(x)$  is independent for every point  $x$  in  $S_d$  if and only if the following condition is satisfied :*

$$(2) \quad \text{for all non-empty } A \subseteq E, \text{ we have } d \geq \frac{\omega(A)}{r(A)}.$$

The proof of Theorem 1.5 is given in Section 2. We first describe some other corollaries of the theorem in the next subsection.

## 1.2 Variants of Arboricity of Matroids

The *arboricity*  $\Upsilon(M)$  of a matroid  $M$  is the minimum number of bases needed to cover all elements of the matroid. Since every base can contain at most  $r(A)$  elements for any  $A \subseteq E$ , the arboricity of a matroid is at least  $\max_{\emptyset \neq A \subseteq E} \frac{|A|}{r(A)}$ . Call this maximum the *maximal density*  $\gamma(M)$ . (Notice that  $M$  is uniformly dense if and only if  $\gamma(M) = |E|/r(E)$ .)

A classical result of Edmonds [3], extending the result for graphs by Nash-Williams [10], guarantees that this lower bound gives the right answer.

**Theorem 1.6 ( Edmonds [3] )**

For a loopless matroid  $M$  we have  $\Upsilon(M) = \lceil \gamma(M) \rceil$ .

The *fractional arboricity*  $\Upsilon_f(M)$  of a matroid  $M$  is defined as follows. To each base  $B$  of  $M$  assign a real value  $x(B) \geq 0$ , such that  $\sum_{B \ni e} x(B) \geq 1$  for all  $e \in E$ . Then  $\Upsilon_f(M)$  is the minimum of  $\sum_{B \in \mathcal{B}(M)} x(B)$  we can obtain under these conditions. Again we have that  $\Upsilon_f(M)$  is at least the maximal density  $\gamma(M)$ , but as has been observed by several authors ( see, e.g., Catlin *et al.* [1] and Scheinerman and Ullman [12, Section 5.4] ), it follows easily from Edmonds' theorem mentioned above that we have equality.

**Proposition 1.7**

For a loopless matroid  $M$  we have  $\Upsilon_f(M) = \gamma(M)$ .

We now define a third kind of arboricity, the *circular arboricity*  $\Upsilon_c(M)$ , introduced by Gonçalves [7]. As before, let  $S_d$  be the circle with circumference  $d$ . Given a matroid  $M$ , we want to map the elements of  $E$  to  $S_d$  so that for every cyclic unit interval  $[x, x + 1]$ , the elements mapped to that cyclic interval form an independent set. Define  $\Upsilon_c(M)$  as the infimum over the values of  $d$  for which such a mapping is possible. Since we assume the matroid to be finite and loopless, it is easy to see that this infimum is actually attained and is a rational number.<sup>1</sup>

The definition of the circular arboricity mimics that of the *circular chromatic number* of a graph. A *stable set* of a graph is a vertex set in which no pair is adjacent. The minimum number of stable sets to cover the vertex set of  $G$  is the *chromatic number*  $\chi(G)$ , the fractional variant is the *fractional chromatic number*  $\chi_f(G)$ , and the circular variant ( the minimum  $d$  such that the vertices of  $G$  can be mapped to  $S_d$  so that the elements mapped to any cyclic unit interval  $[x, x + 1]$  form a stable set ) is the *circular chromatic number*  $\chi_c(G)$ . See, e.g., Zhu [15, 16] for results on this last parameter ( including other ways to define it ).

The following result mimics well-known relations between fractional, circular and integral chromatic number of graphs. For completeness, we give its proof in Section 4.

**Proposition 1.8**

For a loopless matroid  $M$  we have  $\Upsilon_f(M) \leq \Upsilon_c(M) \leq \Upsilon(M)$  and  $\Upsilon(M) = \lceil \Upsilon_c(M) \rceil$ .

It is well-known that the difference between the fractional chromatic number and the integral chromatic number of a graph  $G$  can be arbitrarily large ( see, e.g., Scheinerman and Ullman [12, Chapter 3] ). Because  $\chi(G) = \lceil \chi_c(G) \rceil$ , the same holds for the difference between the fractional chromatic number and the circular chromatic number. It is for that reason somewhat surprising that the fractional and circular arboricity of matroids are always equal.

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<sup>1</sup> We would have the same definition if cyclic intervals were open on both sides or left-open, right-closed. With closed cyclic unit intervals we get the same value for the circular arboricity, but it would be a real infimum in that instance.

**Theorem 1.9**

For a loopless matroid  $M$  we have  $\Upsilon_c(M) = \Upsilon_f(M) = \gamma(M)$ .

This result was conjectured for graphical matroids by Gonçalves [7, Section 3.8]. We give the short derivation from Theorem 1.5 in Section 4.

## 2 Proof of Theorem 1.5

We use the notation and conventions from the first section. Since the weights  $\omega(e)$  and the number  $d$  in the hypothesis of Theorem 1.5 are assumed to be rational, it is clear that we can restrict ourselves to mappings from  $E$  to the rational elements of  $S_d$ . This also shows that the theorem is equivalent to Theorem 2.1 below.

A  $D$ -gon, for a positive integer  $D$ , is the sequence  $(1, 2, \dots, D)$  in cyclic order; in other words: the integer elements of the circle  $S_D$ . For simplicity, we use  $[D] = \{1, \dots, D\}$  for the elements of the  $D$ -gon, but we must remain aware of the cyclic structure of the  $D$ -gon. In particular, for a mapping  $\phi : E \rightarrow [D]$  the cyclic interval  $[\phi(e), \phi(e) + \omega(e)]$  corresponds to the sequence of integers  $(\phi(e), \phi(e) + 1, \dots, \phi(e) + \omega(e) - 1)$ , taken modulo  $D$ .

**Theorem 2.1**

Let  $(M, \omega)$  be a loopless weighted matroid with non-negative integer weights and  $D$  a positive integer. The following statements are equivalent.

- (a) There exists a mapping  $\phi : E \rightarrow [D]$  such that for every  $x \in [D]$ , the set  $\{e \in E \mid x \in [\phi(e), \phi(e) + \omega(e)]\}$  is independent.
- (b) For all  $A \subseteq E$ , we have  $\omega(A) \leq D \cdot r(A)$ .

In the remainder we often use  $A - e$  for  $A \setminus \{e\}$ , and  $A \cup e$  for  $A \cup \{e\}$ .

An essential tool in our proof is the closure operator for matroids. In particular, we will use repeatedly that if  $e \in A \subseteq E$ , then  $e \in \text{cl}(A - e)$  if and only if  $e$  is contained in a circuit of  $A$ . The set  $A$  spans  $E$  if  $\text{cl}(A) = E$ .

**Proof of Theorem 2.1** For a mapping  $\phi : E \rightarrow [D]$  and  $e \in E$ , write  $J_\phi(e) = [\phi(e), \phi(e) + \omega(e)]$ . Recall the notation  $E_\phi(x) = \{e \in E \mid x \in J_\phi(e)\}$ , for any  $x \in [D]$ .

Suppose first that a mapping  $\phi$  satisfying (a) exists. For a set  $A \subseteq E$ , count the pairs  $(a, x)$  with  $a \in A$  and  $x \in J_\phi(a)$  in two ways. Since each  $J_\phi(a)$  contains  $\omega(a)$  elements from  $[D]$ , there are  $\omega(A)$  such pairs. On the other hand, for each  $x \in [D]$  we have that  $E_\phi(x)$  is independent, hence the number of  $a \in A$  with  $x \in J_\phi(a)$  is at most  $r(A)$ . This gives that there are at most  $D \cdot r(A)$  pairs, proving that (b) holds.

So we are left to prove (b)  $\Rightarrow$  (a). For this, let  $D$  satisfy the condition in (b). We prove (a) by induction on  $|E|$ . (It is trivially true if  $|E| = 1$ .)

If there is an  $e \in E$  such that  $\omega(e) = 0$ , then we can remove  $e$  from the matroid and are done by induction. So we can assume  $\omega(e) > 0$  for all  $e \in E$ .

By (b),  $\omega(e) \leq D$  for all  $e \in E$ . Suppose there is an  $e \in E$  with  $\omega(e) = D$ . Let  $M'$  be the contraction  $M/e$ , with rank function  $r'$  and ground set  $E' = E - e$ . Since  $\omega(e') > 0$  for all  $e' \in E$ , (b) guarantees  $r(\{e, e'\}) = 2$  for all  $e' \neq e$ . Thus  $M'$  is loopless. For all  $A' \subseteq E'$ ,

we have  $\omega(A') = \omega(A' + e) - D \leq D \cdot r(A' + e) - D = D \cdot (r'(A') + 1) - D = D \cdot r'(A')$ . So we can apply the induction hypothesis on  $M'$ : there is a mapping  $\phi : E' \rightarrow [D]$  such that  $\{e' \in E' \mid x \in [\phi(e'), \phi(e') + \omega(e')] \}$  is independent for every  $x \in [D]$ . Extend  $\phi$  to  $M$  by setting  $\phi(e) = 1$  (or any other element of  $[D]$ ). It is easy to check that  $\phi$  satisfies (a).

So from now on we assume  $1 \leq \omega(e) \leq D - 1$  for all  $e \in E$ .

Given two mappings  $\phi, \phi'$  of  $E$  to the  $D$ -gon, we say that  $\phi$  is better than  $\phi'$  if for every  $x \in [D]$ ,  $\text{cl}(E_{\phi'}(x)) \subseteq \text{cl}(E_\phi(x))$ . We also say that  $\phi$  is strictly better than  $\phi'$  if the inclusion is strict for some  $x$ ; while  $\phi$  is best possible if no other mapping is strictly better.

Since there are only finitely many mappings to the  $D$ -gon, we can choose a best possible mapping  $\phi$ . Our goal is to prove that  $\phi$  satisfies (a) in the theorem. Assume that this is not the case. So there is some  $x \in [D]$  for which  $E_\phi(x)$  is not independent, i.e.,  $|E_\phi(x)| > r(E_\phi(x))$ , which also means that  $E_\phi(x)$  contains a circuit. Since  $\omega(E) \leq D \cdot r(E)$ , at the same time there must be a point  $x' \in [D]$  for which  $r(E_\phi(x')) < r(E)$ , hence  $E_\phi(x')$  does not span  $E$ .

For an element  $e \in E$ , a push of  $e$  consists of replacing  $\phi(e)$  by  $\phi(e) + 1$  (modulo  $D$ ) (although intuitively it is probably more useful to think of it as replacing the cyclic interval  $[\phi(e), \phi(e) + \omega(e)]$  by  $[\phi(e) + 1, \phi(e) + \omega(e) + 1]$ ). We call  $e$  pushable if  $e$  belongs to a circuit in  $E_\phi(\phi(e))$ . If  $e$  is pushable, then a push of  $e$  always results in a better mapping, since the closure of  $E_\phi(\phi(e))$  does not decrease. Moreover, in that case a push gives a strictly better mapping if adding  $e$  to  $E_\phi(\phi(e) + \omega(e) + 1)$  does increase the closure of that set, i.e., if  $e$  does not belong to a circuit of  $E_\phi(\phi(e) + \omega(e) + 1) \cup e$ . As  $\phi$  is assumed to be best possible, no sequence of pushes should result in a strictly better mapping.

On the other hand, there always are pushable elements. This follows from our earlier observation that for some  $x \in [D]$ , there is a circuit  $C$  in  $E_\phi(x)$ . Going back (in negative direction) along the  $D$ -gon, starting from  $x$ , let  $y$  be the last point for which  $C \subseteq E_\phi(y)$ . (Such a point must exist, since  $\omega(e) \leq D - 1$  for all  $e \in E$ .) By the choice of  $y$ , there exists  $e \in C$  such that  $\phi(e) = y$ ; such an  $e$  is pushable.

From now on we assume that we only push elements that are pushable. From the arguments in the previous paragraphs, there exists an infinite sequence of pushes. We make the sequence of pushes deterministic as follows. Start with some initial ordering  $e_1, \dots, e_m$  of the elements. Every time we push an element, rearrange the ordering by moving the pushed element to the back of the sequence. (So elements towards the end of the ordering have been pushed “more recently” than those towards the beginning.) Whenever we have a choice between pushable elements, we always push the first pushable element according to the ordering at that moment.

Considering the deterministic sequence of pushes thus obtained, we call  $e \in E$  bounded if it is pushed a finite number of times; otherwise it is unbounded. Starting with  $\phi$  this means that after a finite number of pushes we obtain a mapping for which all bounded elements have reached their final position on the  $D$ -gon. Continuing with the sequence, the sequence of mappings eventually must become periodic, say with period  $T$ . Let  $\phi_1, \dots, \phi_T$  be the mappings occurring in this periodic sequence. We analyse the properties of this sequence in some detail.

We first remark that by the definition of pushable and the assumption that  $\phi$  is best possible, each  $\phi_i$  is also best possible. This guarantees the following.

**Claim 1** For all  $i, j$  and all  $x \in [D]$ ,  $\text{cl}(E_{\phi_i}(x)) = \text{cl}(E_{\phi_j}(x))$ .

Let  $E^U$  be the set of unbounded elements and  $E^B$  the bounded ones. As some  $E_{\phi_1}(x)$  do not span  $E$ ,  $E^B$  is non-empty. Also, by our supposition that we are dealing with an infinite sequence of pushes,  $E^U$  is not empty.

Let  $e$  be an element in  $E^B$ . Set  $x_e = \phi_1(e)$ . Since  $e$  has reached its final position by the time we consider the mappings  $\phi_1, \dots, \phi_T$ , we have  $\phi_i(e) = \phi_1(e) = x_e$  for all  $i$ .

**Claim 2** For all  $i$ ,  $e$  does not belong to a circuit of  $E_{\phi_i}(x_e)$ .

Indeed, suppose this is false for some  $i$ . Thus  $e$  is pushable in  $\phi_i$ . Since this holds in each of the (infinitely many) later appearances of  $\phi_i$ , eventually  $e$  becomes the first among the pushable elements in  $\phi_i$ . So  $e$  will eventually be pushed, a contradiction.

By Claim 2, all the pushes of elements from  $E_{\phi_i}(x_e)$ , for any  $i$ , involve circuits that do not contain  $e$ . Using Claim 1 this gives  $\text{cl}(E_{\phi_i}(x_e) - e) = \text{cl}(E_{\phi_j}(x_e) - e)$  for all  $i, j$ . Similarly,  $e \notin \text{cl}(E_{\phi_i}(x_e) - e)$  for all  $i$ .

Now, additionally, let  $f$  be an element in  $E^U$ . Since  $f$  cycles infinitely around the  $D$ -gon, there is a  $j$  such that  $f \in E_{\phi_j}(x_e)$ . But that means trivially that  $\text{cl}((E_{\phi_j}(x_e) \cup f) - e) = \text{cl}(E_{\phi_j}(x_e) - e)$ . Using Claim 1 and the relations above, this gives for all  $i, j$ :

$$\text{cl}((E_{\phi_i}(x_e) \cup f) - e) = \text{cl}((E_{\phi_j}(x_e) \cup f) - e) = \text{cl}(E_{\phi_j}(x_e) - e) = \text{cl}(E_{\phi_i}(x_e) - e).$$

Since this holds for all  $f \in E^U$ , we obtain  $\text{cl}((E_{\phi_i}(x_e) \cup E^U) - e) = \text{cl}(E_{\phi_i}(x_e) - e)$ . As  $e \notin \text{cl}(E_{\phi_i}(x_e) - e)$  for all  $i$ , this gives the following.

**Claim 3** For all  $i$  and  $e \in E^B$ , we have  $e \notin \text{cl}((E_{\phi_i}(x_e) \cup E^U) - e)$ .

Since  $E^U \subseteq (E_{\phi_i}(x_e) \cup E^U) - e$ , this immediately leads to  $e \notin \text{cl}(E^U)$ . But this holds for all  $e \in E^B$ , and so  $E^B \cap \text{cl}(E^U) = \emptyset$ . We have proved the following claim:

**Claim 4**  $\text{cl}(E^U) = E^U$ .

Next we prove our final claim.

**Claim 5** For all  $i$  and  $x \in [D]$ , every circuit  $C$  of  $E_{\phi_i}(x)$  is included in  $E^U$ .

For suppose there is an  $x \in [D]$  and a circuit  $C$  in  $E_{\phi_i}(x)$  such that  $C \cap E^B \neq \emptyset$ . Going back (in negative direction) along the  $D$ -gon, starting from  $x$ , let  $y$  be the last point for which  $C \cap E^B \subseteq E_{\phi_i}(y)$ . (Here we use again that  $\omega(e) \leq D - 1$  for all  $e \in E$ ). By the choice of  $y$ , there exists  $e \in C \cap E^B$  such that  $\phi_i(e) = y$ , i.e.,  $y = x_e$ . Since  $e \in C \subseteq E_{\phi_i}(x_e) \cup E^U$ , that would give  $e \in \text{cl}((E_{\phi_i}(x_e) \cup E^U) - e)$ , contradicting Claim 3.

By Claim 4, the contraction  $M/E^U$  with ground set  $E^B$  is loopless. By Claim 5, we have that  $\phi_1|_{E^B} : E^B \rightarrow [D]$  satisfies condition (a) for the weighted matroid  $(M/E^U, \omega|_{E^B})$ .

Now consider  $M \setminus E^B$ , the submatroid of  $M$  restricted to  $E^U$ , and let  $r^U$  be the rank function of this matroid. Since  $r^U(A) = r(A)$  for all  $A \subseteq E^U$ , we have that  $\omega(A) \leq D \cdot r^U(A)$  for all  $A \subseteq E^U$ . Hence by the induction hypothesis, there exists a mapping  $\phi^U : E^U \rightarrow [D]$  that satisfies (a) for the weighted matroid  $(M \setminus E^B, \omega|_{E^U})$ .

Combine the mappings  $\phi_1|_{E^B}$  and  $\phi'$  to a mapping  $\psi$  of  $E$  to the  $D$ -gon :

$$\psi(e) = \begin{cases} \phi_1|_{E^B}(e), & \text{if } e \in E^B; \\ \phi'(e), & \text{if } e \in E^U. \end{cases}$$

For each point  $x \in [D]$ , the set  $E_\psi(x)$  obtained from this mapping has the property that  $E_\psi(x) \cap E^B$  is independent in  $M/E^U$  and  $E_\psi(x) \cap E^U$  is independent in  $M \setminus E^B$ . That means the whole set  $E_\psi(x)$  is independent in  $M$ , proving that  $\psi$  satisfies condition (a) for the weighted matroid  $(M, \omega)$ . This completes the proof of the theorem.  $\square$

The next results are just Theorems 2.1 and 1.5 in terms of dual matroids.

### Corollary 2.2

*Let  $(M, \omega)$  be a loopless weighted matroid with non-negative integer weights and  $D$  a positive integer. The following statements are equivalent.*

- (a) *There exists a mapping  $\phi : E \rightarrow [D]$  such that for every  $x \in [D]$ , the set  $E_\phi(x)$  spans  $E$ .*
- (b) *For all  $A \subseteq E$ , we have  $\omega(A) \geq D \cdot (r(E) - r(E \setminus A))$ .*

**Proof** This follows easily by applying Theorem 2.1 to the dual matroid  $M^*$  with weight  $\omega^*(e) = D - \omega(e)$ , for all  $e \in E$ .  $\square$

From this we can form the dual version of Theorem 1.5.

### Corollary 2.3

*Let  $(M, \omega)$  be a weighted matroid and  $d$  a positive rational number. There exists a mapping  $\phi : E \rightarrow S_d$  such that  $E_\phi(x)$  spans  $E$  for every point  $x$  in  $S_d$  if and only if the following condition is satisfied: for all  $A \subseteq E$  with  $r(E \setminus A) < r(E)$ , we have  $d \leq \frac{\omega(A)}{r(E) - r(E \setminus A)}$ .*

## 3 Cyclic Orderings of Matroids

The main goal of this section is to prove Theorem 1.4. We also give one corollary.

The following is an equivalent formulation of the theorem.

### Theorem 3.1

*Let  $M$  be a loopless matroid of rank  $r$  and with  $m$  elements such that  $\gcd(r, m) = 1$ . The following statements are equivalent.*

- (a) *There exists a cyclic ordering  $(e_1, \dots, e_m)$  of the elements of  $M$  such that every cyclic interval  $(e_i, \dots, e_{i+r-1})$  of length  $r$  is a base of  $M$ .*
- (b) *For all  $A \subseteq E$ , we have  $r \cdot |A| \leq m \cdot r(A)$ .*

**Proof** Suppose first that a cyclic ordering  $(e_1, \dots, e_m)$  satisfying (a) exists. Let  $B_1, \dots, B_m$  be the bases obtained from that ordering, hence each element of  $M$  appears in  $r$  (cyclically) consecutive bases of  $(B_1, \dots, B_m)$ . So for all  $A \subseteq E$ , we have  $r \cdot |A| = |B_1 \cap A| + |B_2 \cap A| + \dots + |B_m \cap A| \leq m \cdot r(A)$ , proving that (b) holds.

Next assume that  $r \cdot |A| \leq m \cdot r(A)$  for all  $A \subseteq E$ . Setting  $\omega(e) = r$  for all  $e \in E$ , and taking  $D = m$ , we can apply Theorem 2.1 to conclude that there exists a mapping  $\phi : E \rightarrow [m]$  such that for every  $x \in [m]$ , the set  $\{e \in E \mid x \in (\phi(e), \phi(e) + 1, \dots, \phi(e) + r - 1)\}$  is independent. (We use the notation from the paragraph preceding Theorem 2.1.) We prove that if  $\gcd(r, m) = 1$ , this mapping gives a cyclic ordering such that every cyclic interval of length  $r$  forms a base.

Since for all  $e \in E$ , the sequences  $(\phi(e), \dots, \phi(e) + r - 1)$  have the same length  $r$ , it follows immediately that for each cyclic interval  $I_r(x) = (x, x + 1, \dots, x + r - 1)$  of length  $r$  on the  $m$ -gon, the set of elements mapped to  $I_r(x)$  forms an independent set. Notice that since  $r \cdot |E| = m \cdot r(E)$ , the number of elements  $e \in E$  that are mapped to a cyclic interval  $I_r(x)$  on the  $m$ -gon is exactly  $r$ . Since these elements form an independent set, there cannot be more than  $r$ . And if there would be fewer than  $r$  mapped to  $I_r(x)$ , then some other cyclic interval would have more than  $r$  elements, which is also impossible.

We have that  $\phi$  is a mapping of the  $m$ -element set  $E$  to the  $m$ -gon such that every set of  $r$  consecutive points from the  $m$ -gon intersects  $\phi(E)$  in  $r$  points. Suppose that  $x \notin \phi(E)$  for some  $x \in [m]$ . Then the  $r - 1$  points on the  $m$ -gon consecutive to  $x$  contain  $r$  elements of  $\phi(E)$ , hence  $x + r \notin \phi(E)$ . Repeating this argument, we also have  $x + 2r, x + 3r, \dots \notin \phi(E)$ . Since  $\gcd(m, r) = 1$ , this would mean that  $\phi(E)$  is empty, a contradiction. Thus  $x \in \phi(E)$  for all  $x \in [m]$ . It follows that  $\phi$  is a bijection, and hence  $\phi$  corresponds to an ordering of  $E$  along the  $m$ -gon. This is exactly the cyclic ordering we were looking for.  $\square$

An immediate corollary is the following, also conjectured, without the condition  $\gcd(w, m) = 1$ , by Kajitani *et al.* [8].

### Corollary 3.2

*Let  $M$  be a loopless matroid with  $m$  elements. Suppose  $w$  is a positive integer such that  $\gcd(w, m) = 1$  and such that for all  $A \subseteq E$ , we have  $w \cdot |A| \leq m \cdot r(A)$ . There exists a cyclic ordering  $(e_1, \dots, e_m)$  of  $E$  such that every cyclic interval  $(e_i, \dots, e_{i+w-1})$  of  $w$  elements is independent.*

**Proof** Let  $M_w$  be the matroid whose independent sets are the independent sets of  $M$  with at most  $w$  elements. Then  $M_w$  is a matroid of rank  $w$ , and the corollary follows from Theorem 3.1.  $\square$

## 4 Circular Arboricity of Matroids and Related Results

In this final section we prove Theorem 1.9, settling a conjecture made for graphs by Gonçalves [7, page 140]. We first give the short proof of Proposition 1.8. As mentioned earlier, this proof mimics the similar relations for the different types or chromatic number of graphs.

**Proof of Proposition 1.8** Set  $c = \Upsilon(M)$  and take bases  $B_1, \dots, B_c$  covering the elements of  $M$ . By removing multiple occurrences of an element, we find  $c$  disjoint independent sets  $I_1, \dots, I_c$ . Now for each  $e \in E$ , if  $e \in I_i$ , then map  $e$  to the point  $i$  on the circle  $S_c$ . Thus every cyclic unit interval contains exactly one of the independent sets  $I_i$ . This proves  $\Upsilon_c(M) \leq c$ .

Next set  $d = \Upsilon_c(M)$ . Suppose  $\phi : E \rightarrow S_d$  satisfies the requirements for the circular arboricity. By going round the circle, this mapping gives an ordering  $(e_1, \dots, e_m)$  of  $E$  (by taking  $(e_1, \dots, e_m)$  such that  $0 \leq \phi(e_1) \leq \phi(e_2) \leq \dots \leq \phi(e_m) < d$ , braking ties arbitrarily). With each element  $e_i$  we associate an independent set  $I_i = \{e \in E \mid \phi(e) \in [\phi(e_i), \phi(e_i)+1]\}$ . If necessary, add arbitrarily chosen extra elements to extend  $I_i$  to a base  $B_i$ . For  $i = 1, \dots, m$ , let  $x(B_i)$  be equal to the difference between  $\phi(e_{i+1})$  and  $\phi(e_i)$  (measuring by going from  $\phi(e_i)$  to  $\phi(e_{i+1})$  in positive direction), where we take  $e_{m+1} = e_1$ . Note that  $x(B_i)$  is zero if  $\phi(e_i) = \phi(e_{i+1})$ . For all other bases  $B$  of  $M$ , set  $x(B) = 0$ . It is easy to check that for all  $e \in E$ , we have  $\sum_{B \ni e} x(B) \geq 1$ , as well as  $\sum_{B \in \mathcal{B}(M)} x(B) = d$ . This proves  $\Upsilon_f(M) \leq d$ .

Continuing from the previous paragraph, take  $d_0 = \lceil d \rceil$ . For  $i = 1, \dots, d_0 - 1$  set  $I'_i = \{e \in E \mid \phi(e) \in [i-1, i]\}$ , while  $I'_{d_0} = \{e \in E \mid \phi(e) \in [d_0-1, d]\}$ . Then  $I'_1, \dots, I'_{d_0}$  is a collection of independent sets covering  $E$ . We can extend these sets to bases covering  $E$ , showing that  $\lceil \Upsilon_c(M) \rceil \geq \Upsilon(M)$ . Since we saw already  $\Upsilon_c(M) \leq \Upsilon(M)$ , and as  $\Upsilon(M)$  is an integer, we must have  $\Upsilon(M) = \lceil \Upsilon_c(M) \rceil$ .  $\square$

### Theorem 4.1

For a loopless matroid  $M$  we have  $\gamma(M) = \Upsilon_f(M) = \Upsilon_c(M)$ .

**Proof** By Propositions 1.7 and 1.8 it is enough to prove that  $\Upsilon_c(M) \leq \gamma(M)$ . Take positive integers  $P, Q$  such that  $\gamma(M) = \frac{P}{Q}$ . Give weight  $\omega(e) = Q$  to all  $e \in E$ . For all  $A \subseteq E$  we have  $|A| \leq \gamma(M) \cdot r(A)$ , hence  $\omega(A) \leq P \cdot r(A)$ . By Theorem 2.1 there is a mapping  $\phi$  of  $E$  to the  $P$ -gon such that for every  $x \in [P]$ , the set  $\{e \in E \mid x \in [\phi(e), \phi(e) + Q - 1]\}$  is independent. That is equivalent to saying that for every point  $x$  of the  $P$ -gon,  $\{e \in E \mid \phi(e) \in [x, x + Q - 1]\}$  is independent. Now we define the mapping  $\psi : E \rightarrow S_{P/Q}$  by setting  $\psi(e) = \phi(e)/Q$ , for all  $e \in E$ . Then  $\psi$  has the property that for every point  $y$  of  $S_{P/Q}$ , the elements of  $E$  mapped to  $[y, y + 1)$  form an independent set. This shows that  $\Upsilon_c(M) \leq P/Q = \gamma(M)$  and completes the proof.  $\square$

We can use exactly the same idea as in the proof above, but using Corollary 2.2 instead of Theorem 2.1, to obtain the following result.

### Theorem 4.2

Let  $M$  be a matroid and  $d$  a real number such that for all  $A \subseteq E$  with  $r(E \setminus A) < r(e)$ , we have  $d \leq \frac{|A|}{r(E) - r(E \setminus A)}$ . There exists a mapping  $\phi$  of  $E$  to the circle  $S_d$  such that for every point  $x$  of the circle, the set  $\{e \in E \mid \phi(e) \in [x, x + 1]\}$  spans  $E$ .

Because Gonçalves formulated his original conjecture in terms of graphs, we give the corollaries of the last two theorems for the case of graphical matroids. For a graph  $G$ , let  $V(G)$  denote the set of vertices,  $E(G)$  the set of edges, and  $c(G)$  the number of components of  $G$ .

### Corollary 4.3

Let  $G$  be a graph and  $d$  a real number such that for every subgraph  $H$  of  $G$  with at least two vertices, we have  $d \geq \frac{|E(H)|}{|V(H)| - 1}$ . There exists a mapping  $\phi$  of the edge set  $E(G)$  to the circle  $S_d$  such that for every point  $x$  of the circle,  $\{e \in E(G) \mid \phi(e) \in [x, x + 1]\}$  forms an acyclic subgraph of  $G$ .

### Corollary 4.4

Let  $G$  be a connected graph and  $d$  a positive real number such that for every set of edges  $A$  that is a cut in  $G$ , we have  $d \leq \frac{|A|}{c(G - A) - 1}$ . There exists a mapping  $\phi$  of the edge set  $E(G)$  to the circle  $S_d$  such that for every point  $x$ , the set  $\{e \in E(G) \mid \phi(e) \in [x, x + 1]\}$  forms a connected spanning subgraph of  $G$ .

Theorem 4.1 generalises a result of Edmonds [3], while Theorem 4.2 generalises another result of Edmonds [4]. Their graphical versions, Corollaries 4.3 and 4.4, generalise classical results of Nash-Williams [10], and of Nash-Williams [9] and Tutte [13], respectively.

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